

# Lower bounds of the blow-up time of the heat equation in convex domains with local nonlinear boundary conditions

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## Abstract

This paper studies the lower bound of the blow-up time  $T^*$  of the heat equation  $u_t = \Delta u$  with local nonlinear Neumann boundary conditions: The normal derivative  $\partial u / \partial n = u^q$  on part of the boundary  $\Gamma_1$  for some  $q > 1$ , while on the other part  $\partial u / \partial n = 0$ . If  $\Omega \subset \mathbb{R}^N$  is convex, then for any  $\alpha \in (0, \frac{1}{N-1})$ , we obtain a lower bound of  $T^*$  which grows like  $|\Gamma|^{-\alpha}$  as  $|\Gamma_1| \rightarrow 0$ , where  $|\Gamma_1|$  represents the surface area of  $\Gamma_1$ . This significantly improves the previous result  $\left[ \ln(|\Gamma_1|^{-1}) \right]^{2/(N+2)}$  as  $|\Gamma_1| \rightarrow 0$ . Comparing to the previous upper bound of  $T^*$  which grows like  $|\Gamma_1|^{-1}$  as  $|\Gamma_1| \rightarrow 0$ , our result is almost optimal in dimension  $N = 2$ , since  $\alpha$  can be arbitrarily close to 1 in this dimension. In addition, if  $\Gamma_1$  is comparable to a ball in  $\mathbb{R}^{N-1}$  after being straightened, we are able to improve the lower bound of  $T^*$  to grow like  $|\Gamma_1|^{-1/(N-1)}$  for  $N \geq 3$  and  $|\Gamma_1|^{-1} / \ln(|\Gamma_1|^{-1})$  for  $N = 2$  as  $|\Gamma_1| \rightarrow 0$ .

## 1 Introduction

### 1.1 Problem and Results

In this paper,  $\Omega$  represents a bounded open set in  $\mathbb{R}^N$  ( $N \geq 2$ ) with  $C^2$  boundary  $\partial\Omega$ ,  $\Gamma_1$  and  $\Gamma_2$  denote two disjoint relatively open subsets of  $\partial\Omega$  with  $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \partial\Omega$ . Moreover,  $\tilde{\Gamma} \triangleq \overline{\Gamma_1} \cap \overline{\Gamma_2}$  is assumed to be  $C^1$  when being regarded as  $\partial\Gamma_1$  or  $\partial\Gamma_2$ . We study the following problem:

$$\begin{cases} u_t(x, t) = \Delta u(x, t) & \text{in } \Omega \times (0, T], \\ \frac{\partial u}{\partial n}(x, t) = u^q(x, t) & \text{on } \Gamma_1 \times (0, T], \\ \frac{\partial u}{\partial n}(x, t) = 0 & \text{on } \Gamma_2 \times (0, T], \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (1.1)$$

where

$$q > 1, u_0 \in C^1(\overline{\Omega}), u_0(x) \geq 0, u_0(x) \not\equiv 0. \quad (1.2)$$

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The normal derivative on the boundary is understood in the following way: for any  $(x, t) \in \partial\Omega \times (0, T]$ ,

$$\frac{\partial u}{\partial n}(x, t) \triangleq \lim_{h \rightarrow 0^+} Du(x_h, t) \cdot \vec{n}(x) \text{ as long as this limit exists,} \quad (1.3)$$

where  $\vec{n}(x)$  denotes the exterior unit normal vector at  $x$  and  $x_h \triangleq x - h\vec{n}(x)$  for  $x \in \partial\Omega$ .  $\partial\Omega$  being  $C^2$  ensures that  $x_h$  belongs to  $\Omega$  when  $h$  is positive and sufficiently small.

Throughout this paper, we write

$$M_0 = \max_{x \in \bar{\Omega}} u_0(x) \quad (1.4)$$

and denote by  $\Phi$  the fundamental solution to the heat equation:

$$\Phi(x, t) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x|^2}{4t}\right), \quad \forall (x, t) \in \mathbb{R}^N \times (0, \infty). \quad (1.5)$$

Constants such as  $C = C(a, b, \dots)$  will always be positive and finite and depend only on the parameters  $a, b, \dots$ . One should also note that  $C$  may stand for different constants in different places.

The recent paper [19] studied (1.1) systematically and according to it, the local solution and the maximal solution to (1.1) are in the sense of the following two definitions, where (1.6) is imposed to guarantee the uniqueness.

**Definition 1.1** ([19], Definition 1.1). *For any  $T > 0$ , a solution to (1.1) on  $\bar{\Omega} \times [0, T]$  means a function  $u \in C^{2,1}(\Omega \times (0, T]) \cap C(\bar{\Omega} \times [0, T])$  that satisfies (1.1) pointwise and for any  $(x, t) \in \tilde{\Gamma} \times (0, T]$ ,  $\frac{\partial u}{\partial n}(x, t)$  exists and*

$$\frac{\partial u}{\partial n}(x, t) = \frac{1}{2} u^q(x, t). \quad (1.6)$$

**Definition 1.2** ([19], Definition 1.2). *We call*

$$T^* \triangleq \sup\{T \geq 0 : \text{there exists a solution to (1.1) on } \bar{\Omega} \times [0, T]\}$$

*to be the maximal existence time for (1.1). Moreover, a function  $u \in C^{2,1}(\Omega \times (0, T^*)) \cap C(\bar{\Omega} \times [0, T^*))$  is called a maximal solution if  $u|_{\bar{\Omega} \times [0, T]}$  is a solution to (1.1) on  $\bar{\Omega} \times [0, T]$  for any  $T \in (0, T^*)$ .*

Based on these two definitions, [19] showed the following three theorems, in which both upper and lower bounds of  $T^*$  were derived in terms of the surface area  $|\Gamma_1| \triangleq \int_{\Gamma_1} dS$ .

**Theorem 1.3** ([19], Theorem 1.3 and part of Theorem 1.4). *Assuming (1.2) and  $\Gamma_1 \neq \emptyset$ , then the maximal existence time  $T^*$  for (1.1) is positive and finite and there exists a unique maximal solution  $u \in C^{2,1}(\Omega \times (0, T^*)) \cap C(\bar{\Omega} \times [0, T^*))$  to (1.1). Moreover,  $u(x, t) > 0$  for any  $(x, t) \in \bar{\Omega} \times (0, T^*)$  and*

$$\sup_{(x, t) \in \bar{\Omega} \times [0, T^*)} u(x, t) = \infty.$$

**Theorem 1.4** ([19], part of Theorem 1.4). *Assuming (1.2),  $\Gamma_1 \neq \emptyset$  and  $\min_{x \in \bar{\Omega}} u_0(x) > 0$ , let  $T^*$  be the maximal existence time for (1.1), then*

$$T^* \leq \frac{1}{(q-1)|\Gamma_1|} \int_{\Omega} u_0^{1-q}(x) dx. \quad (1.7)$$

**Theorem 1.5** ([19], Theorem 1.5). *Assuming (1.2) and  $\Gamma_1 \neq \emptyset$ , let  $T^*$  be the maximal existence time for*

(1.1), then there exists a constant  $C = C(N, q, \Omega)$  such that

$$T^* \geq C^{-\frac{2}{N+2}} \left[ \ln(|\Gamma_1|^{-1}) - (N+2)(q-1) \ln M_0 - \ln(q-1) - \ln C \right]^{\frac{2}{N+2}}, \quad (1.8)$$

where  $M_0$  is defined as (1.4) and  $C$  remains bounded as  $q \rightarrow 1$ . As a result, no matter  $|\Gamma_1| \rightarrow 0$ ,  $M_0 \rightarrow 0$  or  $q \rightarrow 1$ , we have  $T^* \rightarrow \infty$ .

Theorem 1.3 is a fundamental result. Firstly it asserts that (1.1) does not have global solutions. Secondly if denoting

$$M(t) = \max_{(x,\tau) \in \bar{\Omega} \times [0,t]} u(x,\tau), \quad \forall t \in [0, T^*), \quad (1.9)$$

then the maximal existence time  $T^*$  is just the blow-up time of  $M(t)$ . Since the arguments in this paper focus on the estimate of  $M(t)$ , we will call  $T^*$  to be “the blow-up time” instead of “the maximal existence time” from now on. Theorem 1.4 and Theorem 1.5 together bound the blow-up time  $T^*$  from both above and below. However there is a big gap between them as  $|\Gamma_1| \rightarrow 0$ , since the upper bound grows like  $|\Gamma_1|^{-1}$  while the lower bound is only  $[\ln(|\Gamma_1|^{-1})]^{2/(N+2)}$ .

The purpose of this paper is to improve the lower bound and our results consist of two parts. In the first part, we only assume  $\Omega$  to be convex and then obtain a lower bound which grows like a positive power of  $|\Gamma_1|^{-1}$  as  $|\Gamma_1| \rightarrow 0$ . In the second part, we not only require  $\Omega$  to be convex but also impose a special geometric property on  $\Gamma_1$ , then the lower bound will be further improved.

**Part 1:**  $\Omega$  is assumed to be convex. Under this assumption, we improve the lower bound to a positive power of  $|\Gamma_1|^{-1}$ , which is much larger than the previous result  $[\ln(|\Gamma_1|^{-1})]^{2/(N+2)}$  as  $|\Gamma_1| \rightarrow 0$ . The following theorem is the main result of this paper.

**Theorem 1.6.** *Assuming (1.2) and  $\Gamma_1 \neq \emptyset$ , if  $\Omega$  is convex, then there exists  $C = C(N, \Omega) > 0$  such that for any  $\alpha \in (0, \frac{1}{N-1})$ ,*

$$T^* > \frac{C [1 - (N-1)\alpha] M_0^{1-q}}{q-1} |\Gamma_1|^{-\alpha} - \frac{3q}{10(q-1)}, \quad (1.10)$$

where  $T^*$  is the maximal existence time for (1.1) and  $M_0$  is given by (1.4).

Based on this theorem, if  $N$ ,  $\Omega$ ,  $q$  and  $M_0$  are fixed, then for any  $\alpha \in (0, \frac{1}{N-1})$  and small  $|\Gamma_1|$ ,

$$T^* \geq C(\alpha) |\Gamma_1|^{-\alpha}.$$

In particular when the dimension  $N = 2$ ,  $\alpha$  can be any number in  $(0, 1)$ . Comparing to Theorem 1.4, where the upper bound is in the order of  $|\Gamma_1|^{-1}$  as  $|\Gamma_1| \rightarrow 0$ , this result is almost optimal.

In addition, Theorem 1.6 not only reveals the relation between  $T^*$  and  $|\Gamma_1|^{-1}$ , but also gives the asymptotic behaviours of  $T^*$  when  $q \rightarrow 1$  or  $M_0 \rightarrow 0$ .

- Relation between  $T^*$  and  $q$ . On the one hand, if sending  $q \rightarrow 1$  and fixing other factors, (1.10) implies

$$T^* \geq \frac{C}{q-1} \quad (1.11)$$

when  $|\Gamma_1|$  is small (this assures  $C$  in (1.11) to be positive). On the other hand, the upper bound (1.7) implies

$$T^* \leq \frac{C}{q-1}.$$

Thus  $(q-1)T^*$  is bounded from both below and above when  $q \rightarrow 1$ .

- Relation between  $T^*$  and  $M_0$ . If sending  $M_0 \rightarrow 0$  and fixing other factors, (1.10) implies that

$$T^* \geq C M_0^{1-q}.$$

On the other hand, if the initial data  $u_0$  does not oscillate too much as  $M_0 \rightarrow 0$  (e.g. if  $\max_{\Omega} u_0 \leq C \min_{\Omega} u_0$  for some constant  $C$  as  $M_0 \rightarrow 0$ ), then it follows from (1.7) that

$$T^* \leq C \int_{\Omega} u_0^{1-q} dx \leq C M_0^{1-q}.$$

Hence,  $M_0^{q-1} T^*$  is bounded from both below and above as  $M_0 \rightarrow 0$ .

**Part 2:**  $\Omega$  is still assumed to be convex and  $\Gamma_1$  is imposed a special geometric property called ball-comparable as Definition 4.1. Roughly speaking, a partial boundary  $\Gamma_1$  is called ball-comparable if it is comparable to a ball after being flattened out. The precise description is given in Definition 4.1 and two examples are demonstrated right after the definition. Although Definition 4.1 looks complicated, it includes a large class of partial boundaries. Moreover, the constants  $\Lambda$  and  $S_{f^*}$  mentioned in Definition 4.1 usually have universal bounds when  $|\Gamma_1|$  is small, see Example 4.2 and Example 4.3. Taking advantage of this geometric property, we obtain the following two improved conclusions.

**Theorem 1.7.** *Assuming (1.2) and  $\Gamma_1 \neq \emptyset$ , if  $\Omega \subset \mathbb{R}^N (N \geq 3)$  is convex and  $\Gamma_1$  is ball-comparable with  $\Lambda$  and  $S_{f^*}$  defined in Definition 4.1, then there exists a constant  $C = C(N, \Omega)$  such that*

$$T^* > \frac{C M_0^{1-q}}{S_{f^*} \Lambda (q-1)} |\Gamma_1|^{-\frac{1}{N-1}} - \frac{3q}{10(q-1)}, \quad (1.12)$$

where  $T^*$  is the maximal existence time for (1.1) and  $M_0$  is given by (1.4).

**Theorem 1.8.** *Assuming (1.2) and  $\Gamma_1 \neq \emptyset$ , if  $\Omega \subset \mathbb{R}^2$  is convex and  $\Gamma_1$  is ball-comparable with  $\delta_*$ ,  $\Lambda$  and  $S_{f^*}$  defined in Definition 4.1 and  $\Lambda \delta_* \leq 1/2$ , then there exists a constant  $C = C(\Omega)$  such that*

$$T^* > \frac{C M_0^{1-q}}{S_{f^*} \Lambda (q-1)} |\Gamma_1|^{-1} \left[ \ln \left( \frac{2 S_{f^*}}{|\Gamma_1|} \right) \right]^{-1} - \frac{3q}{10(q-1)}, \quad (1.13)$$

where  $T^*$  is the maximal existence time for (1.1) and  $M_0$  is given by (1.4).

Based on these two theorems, if  $N, \Omega, q$  and  $M_0$  are fixed and the geometric constants  $\Lambda, S_{f^*}$  are bounded above uniformly, then as  $|\Gamma_1| \rightarrow 0$ ,

$$T^* > \begin{cases} C |\Gamma_1|^{-\frac{1}{N-1}}, & N \geq 3, \\ C |\Gamma_1|^{-1} / \ln(|\Gamma_1|^{-1}), & N = 2. \end{cases}$$

In particular, when the dimension  $N = 2$ , this lower bound only differs by  $\ln(|\Gamma_1|^{-1})$  from the upper bound in (1.7) as  $|\Gamma_1| \rightarrow 0$ .

## 1.2 Main Ideas

Historically, many works have been devoted to the heat equation with nonlinear Neumann boundary conditions which are analogous to (1.1) but with  $\Gamma_1 = \partial\Omega$ . More precisely, they study the problem

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = h(x, t) & \text{in } \Omega \times (0, T], \\ \frac{\partial u}{\partial n}(x, t) = H(x, t, u(x, t)) & \text{on } \partial\Omega \times (0, T], \\ u(x, 0) = \psi(x) & \text{in } \Omega, \end{cases} \quad (1.14)$$

where  $h \in C^{\alpha, \alpha/2}(\overline{\Omega} \times [0, T])$ ,  $H \in C(\partial\Omega \times [0, \infty) \times (-\infty, \infty))$  and  $\psi \in C^1(\overline{\Omega})$ . For example, [1, 2, 7, 12] discussed the existence and uniqueness of the solution to (1.14) by various methods and in different spaces. [5, 9, 11, 12, 17, 18] studied the upper bound of the blow-up time. [13, 14, 15] estimated the lower bound of the blow-up time. [6, 7, 9, 12, 17] covered some other topics such as the localization of the blow-up points, the blow-up rate, the asymptotic behaviour near the blow-up points and so on. [3, 8, 10, 16] are books or surveys which summarized the works and methods on various issues.

When concerning the bounds of the blow-up time for the problem (1.14) or (1.1), the upper bound is easier to get, thanks to the idea in [17]. As an heuristic argument, we temporarily assume enough smoothness up to the boundary of the solution  $u$  to (1.14) and consider the energy function

$$E(t) = \int_{\Omega} u(x, t)^{1-q} dx,$$

then

$$\begin{aligned} E'(t) &= (1-q) \int_{\Omega} u^{-q} u_t dx \\ &= (1-q) \int_{\Omega} u^{-q} \Delta u dx \\ &= (1-q) \int_{\partial\Omega} u^{-q} \frac{\partial u}{\partial n} dS(x) - q(q-1) \int_{\Omega} u^{-q-1} |\nabla u|^2 dx \\ &= (1-q) |\partial\Omega| - q(q-1) \int_{\Omega} u^{-q-1} |\nabla u|^2 dx \\ &\leq -(q-1) |\partial\Omega|. \end{aligned} \quad (1.15)$$

This implies that  $E(t)$  decreases at a fixed speed, but  $E(t)$  is always nonnegative, so

$$T^* \leq \frac{E(0)}{(q-1) |\partial\Omega|} = \frac{1}{(q-1) |\partial\Omega|} \int_{\Omega} u_0^{1-q} dx.$$

The upper bound (1.7) is obtained similarly by just replacing  $\partial\Omega$  with  $\Gamma_1$ . Based on this idea, the above results could be proven rigorously by some approximation arguments.

The lower bound of  $T^*$  is much harder to get and very few papers study this. An influential work is [14], where it studies (1.1) with  $\Gamma_1 = \partial\Omega$ . They introduce an energy function

$$E_1(t) = \int_{\Omega} u^{2m}(x, t) dx,$$

for some  $m \geq 2q-2$ . By adopting a technique developed in [13], they derive a first order differential inequality for  $E_1(t)$  and then obtain a lower bound for  $T^*$ . But this method has two limitations.

- Firstly it only works for convex domains.

- Secondly it is not directly applicable to the partial boundary case (1.1) when  $\Gamma_1 \neq \partial\Omega$ . Even after necessary modifications, it can only give a lower bound of  $T^*$  which grows like  $\ln(|\Gamma_1|^{-1})$ .

Thus instead of considering the energy function directly, [19] starts from the Representation formula (see Corollary 3.9 in [19] and note that  $(D\Phi)(x-y, t-\tau) = -D_y[\Phi(x-y, t-\tau)]$  there) as follows: for any  $(x, t) \in \partial\Omega \times [0, T^*)$ ,

$$\begin{aligned} u(x, t) &= 2 \int_{\Omega} \Phi(x-y, t) u_0(y) dy - 2 \int_0^t \int_{\partial\Omega} D_y[\Phi(x-y, t-\tau)] \cdot \vec{n}(y) u(y, \tau) dS(y) d\tau \\ &\quad + 2 \int_0^t \int_{\Gamma_1} \Phi(x-y, t-\tau) u^q(y, \tau) dS(y) d\tau \end{aligned} \quad (1.16)$$

$$\triangleq I + II + III. \quad (1.17)$$

Since  $u$  is the unknown function, we refer  $I$ ,  $II$  and  $III$  to be the constant functional, linear functional and nonlinear functional respectively. If denoting

$$\widetilde{M}(t) = \max_{x \in \partial\Omega} u(x, t),$$

then by estimating the heat kernel  $\Phi$  and applying Holder's inequality, the authors in [19] obtain for any  $t \in [0, T^*)$ ,

$$\widetilde{M}^{N+2}(t) \leq C(1+t^{N/2}) \left[ M_0^{N+2} + \int_0^t \widetilde{M}^{N+2}(\tau) d\tau + |\Gamma_1| \int_0^t \widetilde{M}^{q(N+2)}(\tau) d\tau \right] \quad (1.18)$$

$$\triangleq C(1+t^{N/2})(I' + II' + III'), \quad (1.17')$$

where  $I'$ ,  $II'$  and  $III'$  in (1.17') come from  $I$ ,  $II$  and  $III$  in (1.17) respectively. After this, considering the function

$$E_2(t) \triangleq M_0^{N+2} + \int_0^t \widetilde{M}^{N+2}(\tau) d\tau + |\Gamma_1| \int_0^t \widetilde{M}^{q(N+2)}(\tau) d\tau \quad (1.19)$$

which also blows up at  $T^*$ , it follows from a Gronwall's type estimate that  $T^*$  has a lower bound as (1.8).

Although the lower bound (1.8) has carried a lot of information, it is only comparable to logarithm of  $|\Gamma_1|^{-1}$  as  $|\Gamma_1| \rightarrow 0$ , which is much smaller than the upper bound. So in this paper, we intend to improve the lower bound to be at least a positive power of  $|\Gamma_1|^{-1}$  as  $|\Gamma_1| \rightarrow 0$ . In fact, if (1.17') does not contain the linear functional  $II'$ , then by applying Gronwall's type estimate again, it is not hard to get

$$T^* \geq C |\Gamma_1|^{-2/(Nq+2)}, \quad (1.20)$$

which is a positive power of  $|\Gamma_1|^{-1}$ . This motivates us to get rid of  $II$  in (1.17) since  $II'$  comes from  $II$ . Thanks to the identity (2.3), the convexity of  $\Omega$  can be used to realize this as we will explain in the following. Let  $\Omega$  be convex and  $M(t)$  be defined as (1.9), then for any  $t \in [0, T^*)$  satisfying

$$\max_{x \in \partial\Omega} u(x, t) = M(t), \quad (1.21)$$

there exists  $x^0 \in \partial\Omega$  such that  $u(x^0, t) = M(t)$ . Thus, plugging  $x = x^0$  into (1.16),

$$\begin{aligned} M(t) &\leq 2M_0 \int_{\Omega} \Phi(x^0 - y, t) dy + 2M(t) \int_0^t \int_{\partial\Omega} \left| D_y [\Phi(x^0 - y, t - \tau)] \cdot \vec{n}(y) \right| dS(y) d\tau \\ &\quad + 2M^q(t) \int_0^t \int_{\Gamma_1} \Phi(x^0 - y, t - \tau) dS(y) d\tau. \end{aligned} \quad (1.22)$$

Invoking (2.3) gives

$$\int_0^t \int_{\partial\Omega} \left| D_y [\Phi(x^0 - y, t - \tau)] \cdot \vec{n}(y) \right| dS(y) d\tau = \frac{1}{2} - \int_{\Omega} \Phi(x^0 - y, t) dy.$$

Plugging this identity into (1.22) and simplifying, one has

$$M(t) \int_{\Omega} \Phi(x^0 - y, t) dy \leq M_0 \int_{\Omega} \Phi(x^0 - y, t) dy + M^q(t) \int_0^t \int_{\Gamma_1} \Phi(x^0 - y, t - \tau) dS(y) d\tau. \quad (1.23)$$

If  $M(t)$  is regarded as the unknown function, then the right hand side of (1.23) does not contain the linear functional of  $M(t)$ , which should enable us to increase the lower bound to be a positive power of  $|\Gamma_1|^{-1}$ .

But to continue from (1.23), the energy method will not work, since (1.23) is proved to be true only for the time  $t$  satisfying (1.21). Thus, we need to analyze (1.23) discretely and the first question is what kind of time  $t$  satisfies (1.21)? By the maximum principle, if at some  $t > 0$ , the function  $u$  first reaches some maximum value, then such  $t$  must satisfy (1.23). For example, choosing any  $\lambda_1 > 1$ , if we write  $M_1 = \lambda_1 M_0$  and denote  $T_1$  to be the first time that  $M(t)$  reaches  $M_1$ , then this  $T_1$  satisfies (1.21). Another disadvantage of (1.23) is that although it gets rid of the linear functional of  $M(t)$ , there comes an extra term  $\int_{\Omega} \Phi(x^0 - y, t) dy$  on the left hand side, which decays like  $t^{-N/2}$  when  $t$  becomes large. Hence, to avoid the effect of this decay,  $T_1$  should be kept small.

Taking these restrictions into consideration, we need to come up with some delicate strategies, the detailed arguments will be shown in Section 3. The rough idea is as follows. We will first choose a suitable small  $\lambda_1$  such that there is still a lower bound for  $T_1$ , say  $T_1 \geq 1$ . Then we regard  $u(\cdot, T_1)$  as the “initial data” and repeat the first step and so on. Finally if such process can proceed for  $L$  steps, then this  $L$  is a lower bound of  $T^*$ , since the time in each step has a lower bound 1. So the question boils down to finding a lower bound of the total steps  $L$ .

### 1.3 Future Work

Finally, we want to make some comments on the future work. Since there is a big gap between the lower bound  $|\Gamma_1|^{-1/(N-1)}$  and the upper bound  $|\Gamma_1|^{-1}$  when  $N \geq 3$ , it is desirable to narrow this gap.

- On the one hand, the numerical results given in Section 5 indicate that the blow-up time  $T^*$  is growing like  $|\Gamma|^{-1}$  as  $|\Gamma_1| \rightarrow 0$ . Moreover through the derivation of the lower bound, we bound the term

$$\int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) u^q(y, \tau) dS(y) d\tau$$

in Representation formula (1.16) by the term

$$M^q(t) \int_0^t \int_{\Gamma_1} \Phi(x^0 - y, t - \tau) dS(y) d\tau$$

in (1.23), which loses a lot by enlarging all  $u(y, \tau)$  to be the maximum  $M(t)$ . Thus, it seems possible to improve the lower bound.

- On the other hand, although the numerical results give the upper bound, theoretically the derivation (1.15) for the upper bound also drops a term

$$q(q-1) \int_{\Omega} u^{-q-1} |\nabla u|^2 dx,$$

which should be very large since  $|\nabla u|$  is at least  $u^q$  near the boundary  $\Gamma_1$ . Hence it may also be possible to reduce the upper bound.

- In practice, the domain  $\Omega$  may not be convex everywhere, only the local convexity or concavity is reasonable. Since  $\Gamma_1$  is considered to be a small part of the boundary, it suggests to generalize the results in this paper by only assuming convexity or concavity near  $\Gamma_1$  instead of the whole boundary  $\partial\Omega$ .

## 1.4 Organization

The organization of this paper is as follows. In Section 2, we state several basic results on the heat kernel which will be used later. Section 3 is devoted to demonstrate the main idea and a detailed proof of Theorem 1.6. In Section 4, we first give the definition of the ball-comparable partial boundary and then verify Theorem 1.7 and Theorem 1.8 by the similar ideas in Section 3. Finally Section 5 gives some numerical simulations in dimensions 2 and 3.

## 2 Preliminary results

In the rest of this paper, for any bounded open set  $\Omega$  in  $\mathbb{R}^N$  with  $C^2$  boundary  $\partial\Omega$ , we write

$$B_1 = \sup_{\tau > 0} \sup_{x \in \partial\Omega} \tau^{-\frac{N-1}{2}} \int_{\partial\Omega} e^{-\frac{|x-y|^2}{4\tau}} dS(y). \quad (2.1)$$

It is shown in ([19], Lemma 3.1) that  $B_1$  is a finite positive constant depending only on  $\Omega$  and  $N$ . This section will discuss several results concerning the heat kernel  $\Phi$ .

**Lemma 2.1.** *Let  $\Phi$  be defined as (1.5), then*

$$\int_{\Omega} \Phi(x-y, t) dy - \int_0^t \int_{\partial\Omega} D_y[\Phi(x-y, t-\tau)] \cdot \vec{n}(y) dS(y) d\tau = \frac{1}{2}, \quad \forall x \in \partial\Omega, t > 0. \quad (2.2)$$

*In addition, if  $\Omega$  is convex, then*

$$\int_{\Omega} \Phi(x-y, t) dy + \int_0^t \int_{\partial\Omega} \left| D_y[\Phi(x-y, t-\tau)] \cdot \vec{n}(y) \right| dS(y) d\tau = \frac{1}{2}, \quad \forall x \in \partial\Omega, t > 0. \quad (2.3)$$

*Proof.* For the following problem

$$\begin{cases} u_t(x, t) = \Delta u(x, t) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial n}(x, t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = 1 & \text{in } \Omega, \end{cases} \quad (2.4)$$



it obviously has the unique solution  $u \equiv 1$  on  $\overline{\Omega} \times [0, \infty)$ . As a result, plugging  $u \equiv 1$  into the Representation formula (1.16) (taking  $\Gamma_1 = \emptyset$ ), (2.2) follows.

Now if  $\Omega$  is convex, then  $D_y[\Phi(x - y, t - \tau)] \cdot \vec{n}(y) \leq 0$  for any  $x, y \in \partial\Omega$ . Thus, (2.2) implies (2.3).  $\square$

**Corollary 2.2.** *Let  $F : \partial\Omega \times [0, 1] \rightarrow \mathbb{R}$  be*

$$F(x, t) = \begin{cases} \int_{\Omega} \Phi(x - y, t) dy & \text{for } x \in \partial\Omega, t \in (0, 1], \\ 1/2 & \text{for } x \in \partial\Omega, t = 0, \end{cases}$$

*then  $F$  is continuous on  $\partial\Omega \times [0, 1]$ . As a result,*

$$b_1 \triangleq \min_{\partial\Omega \times [0, 1]} F \quad (2.5)$$

*is a positive constant depending only on  $\Omega$  and the dimension  $N$ .*

*Proof.* Since  $\partial\Omega$  is  $C^2$ , there exists  $C = C(N, \Omega)$  such that  $|(x - y) \cdot \vec{n}(y)| \leq C|x - y|^2$  for any  $x, y \in \partial\Omega$ . Hence for any  $x, y \in \partial\Omega$  and  $\tau > 0$ ,

$$\begin{aligned} \left| D_y[\Phi(x - y, \tau)] \cdot \vec{n}(y) \right| &\leq \frac{C|x - y|^2}{\tau^{N/2+1}} e^{-|x-y|^2/(4\tau)} \\ &= \left[ \frac{C|x - y|^2}{\tau} e^{-|x-y|^2/(8\tau)} \right] \tau^{-N/2} e^{-|x-y|^2/(8\tau)} \\ &\leq C \tau^{-N/2} e^{-|x-y|^2/(8\tau)}. \end{aligned}$$

Thus for any  $(x, t) \in \partial\Omega \times (0, 1]$ ,

$$\begin{aligned} &\left| \int_0^t \int_{\partial\Omega} D_y[\Phi(x - y, t - \tau)] \cdot \vec{n}(y) dS(y) d\tau \right| \\ &= \left| \int_0^t \int_{\partial\Omega} D_y[\Phi(x - y, \tau)] \cdot \vec{n}(y) dS(y) d\tau \right| \\ &\leq C \int_0^t \int_{\partial\Omega} \tau^{-N/2} e^{-|x-y|^2/(8\tau)} dS(y) d\tau \\ &= C \int_0^t \tau^{-1/2} \int_{\partial\Omega} \tau^{-(N-1)/2} e^{-|x-y|^2/(8\tau)} dS(y) d\tau \\ &\leq C B_1 \int_0^t \tau^{-1/2} d\tau = C\sqrt{t}. \end{aligned}$$

Now it follows from (2.1) that

$$\lim_{t \rightarrow 0^+} F(x, t) = 1/2, \quad \forall x \in \partial\Omega \quad (2.6)$$

and this convergence is uniform in  $x$ . Hence,  $F$  is continuous on  $\partial\Omega \times [0, 1]$ .  $\square$

**Lemma 2.3.** *Let  $\Omega$  and  $\Gamma_1$  be the same as in (1.1) and  $\alpha \in (0, \frac{1}{N-1})$ , then there exists  $C = C(N, \Omega, \alpha)$  such that for any  $x \in \partial\Omega$  and  $t \in [0, 1]$ ,*

$$\int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) dS(y) d\tau \leq C |\Gamma_1|^\alpha. \quad (2.7)$$

Actually,  $C$  can be chosen as

$$C = C^* \triangleq \frac{2(B_1 + 1)}{(4\pi)^{N/2} [1 - (N-1)\alpha]}, \quad (2.8)$$

where  $B_1$  is given in (2.1).

*Proof.* Fixing  $(x, t) \in \partial\Omega \times (0, 1]$  and  $\alpha \in (0, \frac{1}{N-1})$ , we denote

$$LHS = \int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) dS(y) d\tau.$$

By a change of variable in  $\tau$ ,

$$\begin{aligned} LHS &= \int_0^t \int_{\Gamma_1} \Phi(x - y, \tau) dS(y) d\tau \\ &= \frac{1}{(4\pi)^{N/2}} \int_0^t \tau^{-N/2} \int_{\Gamma_1} e^{-|x-y|^2/(4\tau)} dS(y) d\tau. \end{aligned} \quad (2.9)$$

For any  $m > 1$ , applying Holder's inequality,

$$\int_{\Gamma_1} e^{-|x-y|^2/(4\tau)} dS(y) \leq \left( \int_{\Gamma_1} e^{-m|x-y|^2/(4\tau)} \right)^{1/m} |\Gamma_1|^{(m-1)/m}. \quad (2.10)$$

Recalling the definition of  $B_1$  in (2.1),

$$\begin{aligned} \int_{\Gamma_1} e^{-m|x-y|^2/(4\tau)} d\tau &= \int_{\Gamma_1} e^{-|x-y|^2/[4(\tau/m)]} d\tau \\ &\leq \left( \frac{\tau}{m} \right)^{(N-1)/2} B_1 \\ &\leq \tau^{(N-1)/2} B_1. \end{aligned}$$

Combining this inequality with (2.10),

$$\begin{aligned} \int_{\Gamma_1} e^{-|x-y|^2/(4\tau)} dS(y) &\leq B_1^{1/m} \tau^{(N-1)/(2m)} |\Gamma_1|^{(m-1)/m} \\ &\leq (B_1 + 1) \tau^{(N-1)/(2m)} |\Gamma_1|^{(m-1)/m}. \end{aligned} \quad (2.11)$$

Plugging (2.11) into (2.9),

$$LHS \leq \frac{B_1 + 1}{(4\pi)^{N/2}} |\Gamma_1|^{(m-1)/m} \int_0^t \tau^{-\frac{N}{2} + \frac{N-1}{2m}} d\tau. \quad (2.12)$$

Now let

$$m = \frac{1}{1 - \alpha},$$

which makes  $(m-1)/m = \alpha$ , then (2.12) becomes

$$LHS \leq \frac{B_1 + 1}{(4\pi)^{N/2}} |\Gamma_1|^\alpha \int_0^t \tau^{\frac{1-(N-1)\alpha}{2} - 1} d\tau. \quad (2.13)$$

Since  $\alpha < \frac{1}{N-1}$ , the right hand side of (2.13) is integrable. Therefore

$$\begin{aligned} LHS &\leq \frac{B_1 + 1}{(4\pi)^{N/2}} |\Gamma_1|^\alpha \frac{2}{1 - (N-1)\alpha} t^{\frac{1-(N-1)\alpha}{2}} \\ &\leq \frac{2(B_1 + 1)}{(4\pi)^{N/2} [1 - (N-1)\alpha]} |\Gamma_1|^\alpha, \end{aligned}$$

where the last inequality is due to  $t \leq 1$ . □

### 3 Lower bounds in the convex domain case

#### 3.1 Heuristic arguments

For the problem (1.1) under the assumption (1.2), [19] derived a lower bound for the blow-up time  $T^*$  as in Theorem 1.5. This section intends to find a larger lower bound for  $T^*$  as  $|\Gamma_1| \rightarrow 0$  when  $\Omega$  is convex. We denote  $M_0$  and  $M(t)$  to be the same as in (1.4) and (1.9).

As having been discussed in Section 1, based on the Representation formula (1.16), one can obtain (1.22) as long as  $t$  satisfies (1.21). Then by taking advantage of the convexity of  $\Omega$ , (1.23) is justified. This section will elaborate the further strategies. Before the complete proof, let us do the first step as a heuristic discussion.

- **Step 1.** Considering any  $\lambda_1 > 1$  to be determined, we write  $M_1 = \lambda_1 M_0$  and define  $T_1$  to be the first time that  $M(t)$  reaches  $M_1$ . By the maximum principle,  $T_1$  satisfies (1.21) and there exists  $x^1 \in \partial\Omega$  such that  $u(x^1, T_1) = M_1$ . The plan is to choose a suitably small  $\lambda_1$  such that  $T_1$  still has a fixed lower bound, say  $T_1 > 1$ . To do so, let us first explore what will happen to  $\lambda_1$  if  $T_1 \leq 1$ .

– **Assuming temporarily that  $T_1 \leq 1$ .** By replacing  $x^0$  and  $t$  by  $x^1$  and  $T_1$  in (1.23),

$$M(T_1) \int_{\Omega} \Phi(x^1 - y, T_1) dy \leq M_0 \int_{\Omega} \Phi(x^1 - y, T_1) dy + M^q(T_1) \int_0^{T_1} \int_{\Gamma_1} \Phi(x^1 - y, T_1 - \tau) dS(y) d\tau.$$

Noticing that  $M(T_1) = M_1$ , so the above inequality becomes

$$(M_1 - M_0) \int_{\Omega} \Phi(x^1 - y, T_1) dy \leq M_1^q \int_0^{T_1} \int_{\Gamma_1} \Phi(x^1 - y, T_1 - \tau) dS(y) d\tau, \quad (3.1)$$

which gives a relation between  $M_0$  and  $M_1$ . Thanks to the assumption  $T_1 \leq 1$ , it follows from Corollary 2.2 and Lemma 2.3 that

$$\int_{\Omega} \Phi(x^1 - y, T_1) dy \geq b_1 \quad (3.2)$$

and for any  $\alpha \in (0, \frac{1}{N-1})$ ,

$$\int_0^{T_1} \int_{\Gamma_1} \Phi(x^1 - y, T_1 - \tau) dS(y) d\tau \leq C^* |\Gamma_1|^\alpha, \quad (3.3)$$

where  $C^*$  is given by (2.8). Plugging (3.2) and (3.3) into (3.1),

$$\frac{M_1 - M_0}{M_1^q} \leq \frac{C^* |\Gamma_1|^\alpha}{b_1}. \quad (3.4)$$

Recalling that  $M_1 = \lambda_1 M_0$ , so

$$\frac{\lambda_1 - 1}{\lambda_1^q} \leq \frac{M_0^{q-1} C^* |\Gamma_1|^\alpha}{b_1}. \quad (3.5)$$

As a summary,  $\lambda_1$  will satisfy (3.5) if  $T_1 \leq 1$ .

Based on this observation, if there exists  $\lambda_1 > 1$  such that

$$\frac{\lambda_1 - 1}{\lambda_1^q} = \frac{2M_0^{q-1} C^* |\Gamma_1|^\alpha}{b_1} > \frac{M_0^{q-1} C^* |\Gamma_1|^\alpha}{b_1}, \quad (3.6)$$

then we pick up this  $\lambda_1$  and therefore  $T_1$  has to be greater than 1. This is Step 1.

After the first step, we can regard  $u(\cdot, T_1)$  as the new initial data and repeat the process in Step 1. Continuing in this way until (3.6) no longer has a solution  $\lambda_1 > 1$ . Then the total steps will be a lower bound for  $T^*$ , since the time spent in each step is at least 1. In the following, we translate the above idea into an induction and introduce some notations which will be used in the proof of Theorem 1.6.

- **Step 0.** Let  $p_0 = 1$ ,  $T_0 = 0$  and define  $M_0$ ,  $M(t)$  as in (1.4) and (1.9).
- **Step k.** Suppose  $k \geq 1$  and  $p_{k-1}$ ,  $M_{k-1}$  and  $T_{k-1}$  have been defined, we intend to construct  $p_k$ ,  $M_k$  and  $T_k$ . For any  $\lambda_k > 1$  to be determined, we choose  $p_k = \lambda_k p_{k-1}$  and  $M_k = \lambda_k M_{k-1}$ . Then we define  $T_k$  to be the first time that the function  $M(t)$  reaches  $M_k$  and denote  $\tilde{T}_k \triangleq T_k - T_{k-1}$  to be the time spent in this step. Thanks to the maximum principle,  $T_k$  satisfies (1.21) (Note that  $T_0$  may not satisfy (1.21), but for any  $1 \leq j \leq k$ ,  $T_j$  does). By a similar argument as that in Step 1, in order for  $\tilde{T}_k > 1$ , it suffices to pick up  $\lambda_k > 1$  such that a similar equation as (3.6) holds. Hence, if that equation has a solution greater than 1, then we define  $\lambda_k$  to be that solution and proceed to the next step. Otherwise if that equation does not have a solution greater than 1, then we stop the construction.

So far, this strategy looks promising with only two challenges left. First how to recognize whether (3.6) has a solution  $\lambda_1 > 1$ ? Second how to estimate the number of the total steps? The first question is simple, see Lemma 3.1. The second question is harder to see. We will answer this question and show all the details in the proofs of Theorem 1.6 and Lemma 3.2. In the rest of this paper, for any  $q > 1$ , we denote

$$E_q = (q-1)^{q-1}/q^q. \quad (3.7)$$

By elementary calculus,

$$\frac{1}{qe} < E_q < \min \left\{ \frac{1}{q}, \frac{1}{(q-1)e} \right\} < 1. \quad (3.8)$$

**Lemma 3.1.** For any  $q > 1$ , we write  $E_q$  as in (3.7) and define  $g : (1, \infty) \rightarrow \mathbb{R}$  by

$$g(t) = \frac{t-1}{t^q}, \quad (3.9)$$

then the following two claims hold.

- (1) For any  $y \in (0, E_q]$ , there exists unique  $t \in (1, \frac{q}{q-1}]$  such that  $g(t) = y$ .
- (2) For any  $y > E_q$ , there does not exist  $t > 1$  such that  $g(t) = y$ .

*Proof.* By elementary calculations,  $g$  is strictly increasing on the interval  $(1, \frac{q}{q-1}]$  and strictly decreasing on the interval  $[\frac{q}{q-1}, \infty)$ . Thus, it reaches the maximum at  $t = q/(q-1)$ . Noticing that

$$g\left(\frac{q}{q-1}\right) = \frac{(q-1)^{q-1}}{q^q} = E_q,$$

then the claims (1) and (2) follow directly.  $\square$

### 3.2 Proof of Theorem 1.6

Now we start to prove the main result of this paper.

**Proof of Theorem 1.6.** Following the strategy and the notations discussed above, a complete proof will be given here. First of all, we intend to construct three strictly increasing sequences  $\{p_k\}_{k \geq 0}$ ,  $\{M_k\}_{k \geq 0}$  and  $\{T_k\}_{k \geq 0}$  by induction.

- **Step 0.** Let  $p_0 = 1$ ,  $T_0 = 0$  and define  $M_0$ ,  $M(t)$  as in (1.4) and (1.9).
- **Step k.** Suppose  $k \geq 1$  and  $\{p_j\}_{0 \leq j \leq k-1}$ ,  $\{M_j\}_{0 \leq j \leq k-1}$ ,  $\{T_j\}_{0 \leq j \leq k-1}$  have been defined, we intend to construct  $p_k$ ,  $M_k$  and  $T_k$  in this step. Considering any  $\lambda_k > 1$  to be determined, we choose

$$p_k = \lambda_k p_{k-1}, \quad M_k = \lambda_k M_{k-1}. \quad (3.10)$$

Moreover, we define  $T_k$  to be the first time that  $M(t)$  reaches  $M_k$  and denote

$$\tilde{T}_k \triangleq T_k - T_{k-1} \quad (3.11)$$

to be the time spent in this step.

By the maximum principle, there exists  $x^k \in \partial\Omega$  such that  $u(x^k, T_k) = M_k$ , so  $T_k$  satisfies (1.21). Applying the Representation formula (A.1) with  $T = T_{k-1}$  and  $(x, t) = (x^k, \tilde{T}_k)$ ,

$$\begin{aligned} u(x^k, T_k) &= 2 \int_{\Omega} \Phi(x^k - y, \tilde{T}_k) u(y, T_{k-1}) dy \\ &\quad - 2 \int_0^{\tilde{T}_k} \int_{\partial\Omega} D_y [\Phi(x^k - y, \tilde{T}_k - \tau)] \cdot \vec{n}(y) u(y, T_{k-1} + \tau) dS(y) d\tau \\ &\quad + 2 \int_0^{\tilde{T}_k} \int_{\Gamma_1} \Phi(x^k - y, \tilde{T}_k - \tau) u^q(y, T_{k-1} + \tau) dS(y) d\tau. \end{aligned} \quad (3.12)$$

As a result,

$$\begin{aligned} M_k &\leq 2M_{k-1} \int_{\Omega} \Phi(x^k - y, \tilde{T}_k) dy \\ &\quad + 2M_k \int_0^{\tilde{T}_k} \int_{\partial\Omega} \left| D_y [\Phi(x^k - y, \tilde{T}_k - \tau)] \cdot \vec{n}(y) \right| dS(y) d\tau \\ &\quad + 2M_k^q \int_0^{\tilde{T}_k} \int_{\Gamma_1} \Phi(x^k - y, \tilde{T}_k - \tau) dS(y) d\tau. \end{aligned} \quad (3.13)$$

Since  $\Omega$  is convex, it follows from (2.3) that

$$\int_0^{\tilde{T}_k} \int_{\partial\Omega} \left| D_y [\Phi(x^k - y, \tilde{T}_k - \tau)] \cdot \vec{n}(y) \right| dS(y) d\tau = \frac{1}{2} - \int_{\Omega} \Phi(x^k - y, \tilde{T}_k) dy.$$

Plugging this identity into (3.13) and simplifying,

$$(M_k - M_{k-1}) \int_{\Omega} \Phi(x^k - y, \tilde{T}_k) dy \leq M_k^q \int_0^{\tilde{T}_k} \int_{\Gamma_1} \Phi(x^k - y, \tilde{T}_k - \tau) dS(y) d\tau. \quad (3.14)$$

– **Now let us temporarily assume  $\tilde{T}_k \leq 1$  and check what will happen to  $\lambda_k$ .** Thanks to this assumption, it follows from Corollary 2.2 and Lemma 2.3 that

$$\int_{\Omega} \Phi(x^k - y, \tilde{T}_k) dy \geq b_1 \quad (3.15)$$

and for any  $\alpha \in (0, \frac{1}{N-1})$ ,

$$\int_0^{\tilde{T}_k} \int_{\Gamma_1} \Phi(x^k - y, \tilde{T}_k - \tau) dS(y) d\tau \leq C^* |\Gamma_1|^\alpha, \quad (3.16)$$

where  $C^*$  is given by (2.8). Plugging (3.15) and (3.16) into (3.14),

$$\frac{M_k - M_{k-1}}{M_k^q} \leq \frac{C^* |\Gamma_1|^\alpha}{b_1}. \quad (3.17)$$

Recalling that  $M_k = \lambda_k M_{k-1}$ , then

$$\frac{\lambda_k - 1}{\lambda_k^q M_{k-1}^{q-1}} \leq \frac{C^* |\Gamma_1|^\alpha}{b_1}.$$

It follows from the induction and (3.10) that  $M_{k-1} = p_{k-1} M_0$ , therefore

$$\frac{\lambda_k - 1}{\lambda_k^q p_{k-1}^{q-1}} \leq \frac{M_0^{q-1} C^* |\Gamma_1|^\alpha}{b_1}. \quad (3.18)$$

In summary,  $\lambda_k$  will satisfy (3.18) if  $\tilde{T}_k \leq 1$ .

Based on this observation, if  $\lambda_k > 1$  such that

$$\frac{\lambda_k - 1}{\lambda_k^q p_{k-1}^{q-1}} = \delta_1, \quad (3.19)$$

where

$$\delta_1 \triangleq \frac{2M_0^{q-1} C^* |\Gamma_1|^\alpha}{b_1}, \quad (3.20)$$

then  $\tilde{T}_k$  has to be greater than 1. But when does (3.19) possess a solution  $\lambda_k > 1$ ? According to Lemma 3.1, there are two situations.

**(Continue):  $0 < \mathbf{p}_{k-1}^{q-1} \delta_1 \leq \mathbf{E}_{\mathbf{q}}$ .** In this case, there exists unique  $\lambda_k \in (1, \frac{q}{q-1}]$  such that (3.19) holds. Then we pick up such  $\lambda_k$  and proceed to the next step.

**(Stop):**  $\mathbf{p}_{k-1}^{q-1} \delta_1 > \mathbf{E}_q$ . In this case, there does not exist  $\lambda_k > 1$  such that (3.19) holds. Then we stop the construction at this step. That is to say, we stop the construction at  $p_{k-1}, M_{k-1}, T_{k-1}$ .

Now the question is how many steps can we proceed? Noticing  $\delta_1$  in (3.20) is a fixed number, so the sequence  $\{p_k\}_{k \geq 0}$  is determined by  $p_0 = 1$ , (3.19) and the relation  $p_k = \lambda_k p_{k-1}$ . As a result, the construction of  $\{p_k\}_{k \geq 0}$  actually is independent of  $\{T_k\}_{k \geq 0}$  and  $\{M_k\}_{k \geq 0}$ . Therefore, in order to calculate the total steps for this induction, it suffices to focus on  $\{p_k\}_{k \geq 0}$ . Then it follows from Lemma 3.2 that this procedure must stop in finite steps and if it stops at  $p_L$ , then

$$L > \frac{1}{10(q-1)} \left( \frac{1}{\delta_1} - 3q \right).$$

As a result, we conclude that

$$T^* > \frac{1}{10(q-1)} \left( \frac{1}{\delta_1} - 3q \right), \quad (3.21)$$

since the time spent in each step is at least 1. Now plugging (3.20) into (3.21),

$$T^* > \frac{b_1 M_0^{1-q}}{20(q-1) C^* |\Gamma_1|^\alpha} - \frac{3q}{10(q-1)}. \quad (3.22)$$

Recalling the expression of  $C^*$  in (2.8), we can rewrite (3.22) to be

$$T^* > \frac{C [1 - (N-1)\alpha] M_0^{1-q}}{(q-1) |\Gamma_1|^\alpha} - \frac{3q}{10(q-1)},$$

where

$$C = \frac{(4\pi)^{N/2} b_1}{40(B_1 + 1)}$$

is a constant depending only on  $\Omega$  and  $N$ . □

**Lemma 3.2.** *Given  $q > 1$  and  $\delta_1 > 0$ , we denote  $E_q$  as (3.7) and construct a sequence  $\{p_k\}_{k \geq 0}$  by the following procedure.*

- **Step 0.**  $p_0 \triangleq 1$ .
- **Step k.** Suppose  $k \geq 1$  and  $\{p_j\}_{0 \leq j \leq k-1}$  have been constructed, then there are two situations according to Lemma 3.1.

**(Continue):**  $0 < \mathbf{p}_{k-1}^{q-1} \delta_1 \leq \mathbf{E}_q$ . In this case, let  $\lambda_k$  be the unique number such that (3.23) holds.

$$\lambda_k \in \left( 1, \frac{q}{q-1} \right] \quad \text{and} \quad \frac{\lambda_k - 1}{\lambda_k^q} = p_{k-1}^{q-1} \delta_1. \quad (3.23)$$

Then we define

$$p_k = \lambda_k p_{k-1} \quad (3.24)$$

and continue to the next step.

**(Stop):**  $\mathbf{p}_{k-1}^{q-1} \delta_1 > \mathbf{E}_q$ . In this case, there does not exist  $\lambda_k$  such that (3.23) holds. Then we do not define  $p_k$  and stop the construction at this step. That is to say, the procedure of constructing the sequence stops at  $p_{k-1}$ .

Then we claim

(1) This procedure must stop after finite steps.

(2) If this procedure stops at  $p_L$ , then

$$L > \frac{1}{10(q-1)} \left( \frac{1}{\delta_1} - 3q \right). \quad (3.25)$$

*Proof.* (1) We use proof by contradiction. If this procedure could proceed forever, let  $g$  be given as (3.9) and then for any  $k \geq 1$ , it follows from (3.23) and (3.24) that

$$\frac{g(\lambda_{k+1})}{g(\lambda_k)} = \frac{p_k^{q-1} \delta_1}{p_{k-1}^{q-1} \delta_1} = \lambda_k^{q-1} > 1. \quad (3.26)$$

Thus  $\{\lambda_k\}$  is a strictly increasing sequence, since  $g$  is strictly increasing over the interval  $(1, \frac{q}{q-1}]$ . Combining this fact with (3.26), for any  $k \geq 1$ ,

$$\frac{g(\lambda_{k+1})}{g(\lambda_k)} = \lambda_k^{q-1} \geq \lambda_1^{q-1}. \quad (3.27)$$

Taking advantage of (3.23) again and noticing  $p_0 = 1$ , we have  $\lambda_1 > 1$  and

$$\frac{\lambda_1 - 1}{\lambda_1^q} = \delta_1,$$

so  $\lambda_1 > 1 + \delta_1$ . As a result, it follows from (3.27) that

$$\frac{g(\lambda_{k+1})}{g(\lambda_k)} \geq (1 + \delta_1)^{q-1}.$$

This shows that  $g(\lambda_k)$  increases at least  $(1 + \delta_1)^{q-1}$  every step, which contradicts to the fact that  $g$  is a bounded function on  $(1, \frac{q}{q-1}]$ . Hence the procedure must stop in finite steps.

(2) According to the conclusion in (1), it can be assumed that this procedure stops at  $p_L$  for some  $0 \leq L < \infty$ . Then there are two situations.

- **Case 1:**  $\delta_1 > \mathbf{E}_q$ . In this case, it follows from (3.8) that

$$\frac{1}{\delta_1} < \frac{1}{E_q} < 3q,$$

so the right hand side of (3.25) is negative. Thus (3.25) holds automatically.

- **Case 2:**  $0 < \delta_1 \leq \mathbf{E}_q$ . In this case, the construction will at least proceed to  $\lambda_1 > 1 + \delta_1$ . So

$$L \geq 1, \quad p_{L-1}^{q-1} \delta_1 \leq E_q \quad \text{and} \quad p_L^{q-1} \delta_1 > E_q.$$

Then for any  $1 \leq k \leq L$ , it follows from (3.23) and (3.24) that

$$\frac{\lambda_k - 1}{\lambda_k} = p_k^{q-1} \delta_1, \quad (3.28)$$

which implies

$$\lambda_k = \frac{1}{1 - p_k^{q-1} \delta_1}. \quad (3.29)$$



Invoking (3.24) again,

$$p_{k-1} = p_k(1 - p_k^{q-1}\delta_1).$$

Raising both sides of the above equality to the power  $q-1$  and multiplying by  $\delta_1$ ,

$$p_{k-1}^{q-1}\delta_1 = p_k^{q-1}\delta_1(1 - p_k^{q-1}\delta_1)^{q-1}.$$

Now let  $x_k = p_k^{q-1}\delta_1$ , then

$$x_{k-1} = x_k(1 - x_k)^{q-1}, \quad \forall 1 \leq k \leq L. \quad (3.30)$$

Moreover,

$$x_0 = \delta_1, \quad x_{L-1} \leq E_q \quad \text{and} \quad x_L > E_q.$$

Since (3.23) implies that  $\lambda_L \leq q/(q-1)$ , then it follows from (3.28) that

$$x_L = p_L^{q-1}\delta_1 = \frac{\lambda_L - 1}{\lambda_L} \leq \frac{1}{q}.$$

Noticing that the right hand side of (3.30) is a nonlinear function in  $x_k$ , it is better to consider the “reversed” relation of (3.30). Thus, we define a new sequence  $\{y_k\}_{0 \leq k \leq L}$  as the following way:  $y_0 \triangleq \min\{1/2, E_q\}$  and

$$y_k \triangleq y_{k-1}(1 - y_{k-1})^{q-1}, \quad \forall 1 \leq k \leq L. \quad (3.31)$$

In addition, define  $h : (0, 1) \rightarrow \mathbb{R}$  by

$$h(t) = t(1 - t)^{q-1}.$$

It is easy to see that  $h$  is strictly increasing on  $(0, 1/q]$  and strictly decreasing on  $[1/q, 1)$ . As a result, it follows from  $0 < y_0 < x_L \leq 1/q$  that

$$y_1 = h(y_0) < h(x_L) = x_{L-1}.$$

Keep doing this, we get  $y_k < x_{L-k}$  for any  $0 \leq k \leq L$ . Especially when  $k = L$ ,  $y_L < x_0 = \delta_1$ .

Since  $\{y_k\}$  is a decreasing positive sequence and  $y_0 \leq 1/2$ , then  $y_k \leq 1/2$  for any  $0 \leq k \leq L$ . As a result, it follows from (3.31) and the mean value theorem that for any  $1 \leq k \leq L$ ,

$$y_k \geq y_{k-1}[1 - 2(q-1)y_{k-1}]. \quad (3.32)$$

Recalling (3.8) again,

$$y_{k-1} \leq y_0 \leq E_q < \frac{1}{(q-1)e},$$

so

$$1 - 2(q-1)y_{k-1} > 1 - \frac{2}{e} > \frac{1}{5}.$$

Therefore, we can take reciprocal in (3.32) to obtain for any  $1 \leq k \leq L$ ,

$$\begin{aligned} \frac{1}{y_k} &\leq \frac{1}{y_{k-1} [1 - 2(q-1)y_{k-1}]} \\ &= \frac{1}{y_{k-1}} + \frac{2(q-1)}{1 - 2(q-1)y_{k-1}} \\ &< \frac{1}{y_{k-1}} + 10(q-1). \end{aligned} \tag{3.33}$$

Summing up (3.33) for  $k$  from 1 to  $L$ , then

$$\frac{1}{y_L} < \frac{1}{y_0} + 10(q-1)L. \tag{3.34}$$

Since  $y_L < \delta_1$  and

$$y_0 = \min\{1/2, E_q\} > \frac{1}{3q},$$

it follows from (3.34) that

$$\frac{1}{\delta_1} < 3q + 10(q-1)L.$$

Thus,

$$L > \frac{1}{10(q-1)} \left( \frac{1}{\delta_1} - 3q \right).$$

□

## 4 Improvements under further geometric assumptions

### 4.1 Definition of the ball-comparable partial boundary

In this section, we still assume the convexity of  $\Omega$  but add another geometric assumption on  $\Gamma_1$ , which will be used to improve the lower bounds further. Roughly speaking,  $\Gamma_1$  is required to be comparable to a ball in  $\mathbb{R}^{N-1}$  after being flattened out. The precise description is given in Definition 4.1 in a similar way as the definition of a  $C^1$  boundary (e.g. see Appendix C.1 in [4]).

For the convenience of the statement, we denote  $x$  and  $\tilde{x}$  to be the general points in  $\mathbb{R}^N$  and  $\mathbb{R}^{N-1}$  respectively. Moreover, for any point  $x \in \mathbb{R}^N$ , we write  $\bar{x} \in \mathbb{R}^{N-1}$  to be the first  $N-1$  components of  $x$ . In addition,  $B(x, r)$  represents the ball in  $\mathbb{R}^N$  with center  $x$  and radius  $r$  while  $\tilde{B}(\tilde{x}, r)$  denotes the ball in  $\mathbb{R}^{N-1}$  with center  $\tilde{x}$  and radius  $r$ .

**Definition 4.1** (ball-comparable partial boundary). *Let  $\Omega \subset \mathbb{R}^N$  be an open bounded set with  $C^1$  boundary  $\partial\Omega$  and  $\Gamma_1$  be a relatively open subset of  $\partial\Omega$ , then  $\Gamma_1$  is called ball-comparable if (upon relabeling and reorienting the coordinates axes if necessary) there exists  $y^* \in \Gamma_1$ ,  $r_* > 0$ ,  $\delta_* > 0$ ,  $\Lambda \geq 1$  and a  $C^1$  function  $f^* : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$  such that the following two conditions hold.*

(1)  $\Gamma_1 \subset B(y^*, r_*) \cap \partial\Omega$  and

$$\Omega \cap B(y^*, r_*) = \{x \in B(y^*, r_*) : x_N > f^*(\bar{x})\}.$$

(2)  $\Gamma_1$  is between the graph of  $f^*$  over the set  $\tilde{B}(\bar{y}^*, \delta_*)$  and the set  $\tilde{B}(\bar{y}^*, \Lambda\delta_*)$ . That is,

$$\{(\tilde{y}, f^*(\tilde{y})) : \tilde{y} \in \tilde{B}(\bar{y}^*, \delta_*)\} \subset \Gamma_1 \subset \{(\tilde{y}, f^*(\tilde{y})) : \tilde{y} \in \tilde{B}(\bar{y}^*, \Lambda\delta_*)\}.$$

$\delta_*$ ,  $\Lambda$ ,  $f^*$  are called the comparable radius, the comparable multiple and the defining function of  $\Gamma_1$  respectively. Finally, we define

$$S_{f^*} = \sup_{\tilde{y} \in \tilde{B}(\bar{y}^*, \Lambda\delta_*)} \sqrt{1 + |Df^*(\tilde{y})|^2}. \quad (4.1)$$

Although this definition looks complicated, it actually includes a large class of partial boundaries. As heuristic examples, we give two simplest kinds of partial boundaries in  $\mathbb{R}^2$  which satisfy this definition.

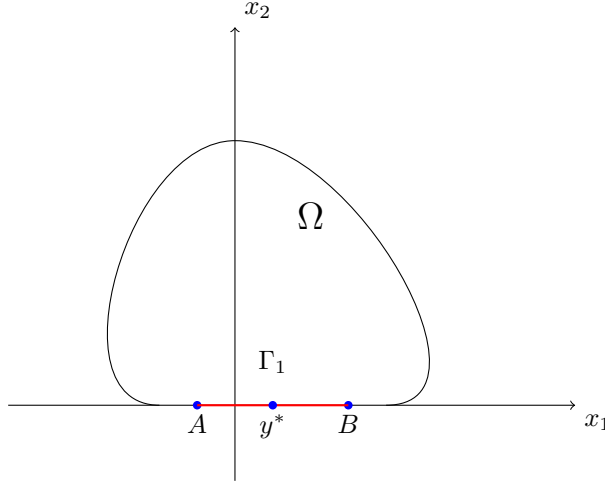


Figure 1: Example 4.2

**Example 4.2.** As shown in Figure 1,  $\Omega \subset \mathbb{R}^2$  has a flat part on  $x_1$ -axis and  $\Gamma_1$  lies on this flat part. More precisely, let  $\Gamma_1$  be the open interval between  $A$  and  $B$ , then it is ball-comparable.

*Proof.* Let  $y^*$  be the middle point of  $A$  and  $B$ , we define  $r_* = |y^* - A|$  (or equivalently  $r_* = |y^* - B|$ ) and choose the defining function to be

$$f^*(x_1) \triangleq x_1, \quad \forall x_1 \in \mathbb{R}.$$

Then it is easy to see that condition (1) in Definition 4.1 is satisfied. Next, we define  $\delta_* = |y^* - A|$  (or equivalently  $\delta_* = |y^* - B|$ ) and choose  $\Lambda = 1$ , then the condition (2) in Definition 4.1 is satisfied. Thus,  $\Gamma_1$  is ball-comparable.

Finally, due to the choice of  $f^*$ , it is obvious that  $S_{f^*} = \sqrt{2}$ . Hence, the bounds on  $\Lambda$  and  $S_{f^*}$  are uniform constants (independent of the choice of  $A$  and  $B$ ).  $\square$

**Example 4.3.** As shown in Figure 2,  $\Omega \subset \mathbb{R}^2$  is the unit circle and  $\Gamma_1$  is a relatively open arc of  $\partial\Omega$ . More precisely, let  $\Gamma_1$  be the shorter open arc between  $A$  and  $B$ , where  $A$  and  $B$  are symmetric with respect to  $x_2$ -axis and

$$A = (\cos(\theta_1), \sin(\theta_1)), \quad B = (\cos(\theta_2), \sin(\theta_2)), \quad \frac{5}{4}\pi < \theta_1 < \theta_2 < \frac{7}{4}\pi.$$

Then this  $\Gamma_1$  is ball-comparable.

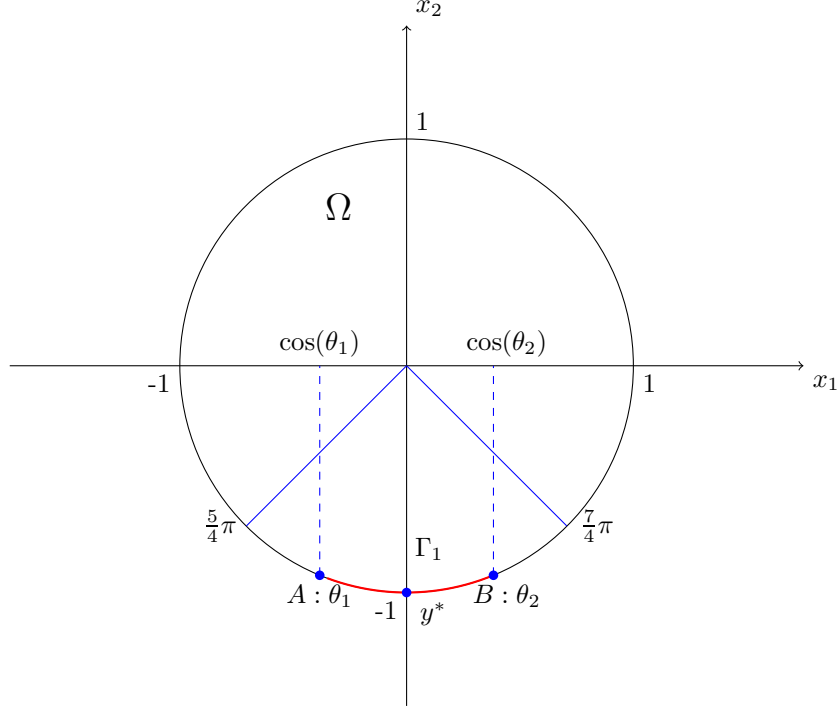


Figure 2: Example 4.3

*Proof.* Since  $A$  and  $B$  are symmetric with respect to  $x_2$ -axis, we choose  $y^* = (0, -1)$  and  $r^* = |y^* - A|$  (or equivalently  $r^* = |y^* - B|$ ). In addition, we choose the defining function  $f^*$  to be

$$f^*(x_1) = \begin{cases} -\sqrt{1 - x_1^2}, & |x_1| \leq 0.8, \\ C^1 \text{ extension}, & |x_1| > 0.8. \end{cases} \quad (4.2)$$

Then it is easy to see that the condition (1) in Definition 4.1 is satisfied. Next we define  $\delta_* = \cos(\theta_2)$  (or equivalently  $\delta_* = -\cos(\theta_1)$ ) and choose  $\Lambda = 1$ , then the condition (2) in Definition 4.1 is satisfied. Thus,  $\Gamma_1$  is ball-comparable.

Finally noticing that  $|\cos(5\pi/4)| = |\cos(7\pi/4)| < 0.8$ , it follows from (4.1) and (4.2) that

$$S_{f^*} \leq \sup_{|x_1| \leq 0.8} \sqrt{1 + (Df^*(x_1))^2} = \sup_{|x_1| \leq 0.8} \frac{1}{\sqrt{1 - x_1^2}} = \frac{5}{3}.$$

Hence, the bounds on  $\Lambda$  and  $S_{f^*}$  are uniform constants (independent of the choice of  $A$  and  $B$ ).  $\square$

Based on Definition 4.1, it is also evident that the surface area  $|\Gamma_1|$  is comparable to  $\delta_*^{N-1}$ . In fact, let  $\Gamma_1$  be ball-comparable and let  $\pi_{N-1}$  denote the volume of the unit ball in  $\mathbb{R}^{N-1}$ , then the condition (2) in Definition 4.1 implies the following two estimates:

$$\begin{aligned} |\Gamma_1| &= \int_{\Gamma_1} dS(y) \geq \int_{\tilde{B}(\overline{y^*}, \delta_*)} \sqrt{1 + |Df^*(\tilde{y})|^2} d\tilde{y} \\ &\geq \int_{\tilde{B}(\overline{y^*}, \delta_*)} d\tilde{y} = \pi_{N-1} \delta_*^{N-1} \end{aligned}$$

and

$$\begin{aligned}
|\Gamma_1| &= \int_{\Gamma_1} dS(y) \leq \int_{\tilde{B}(\overline{y^*}, \Lambda\delta_*)} \sqrt{1 + |Df^*(\tilde{y})|^2} d\tilde{y} \\
&\leq S_{f^*} \int_{\tilde{B}(\overline{y^*}, \Lambda\delta_*)} d\tilde{y} \\
&= \pi_{N-1} S_{f^*} \Lambda^{N-1} \delta_*^{N-1}.
\end{aligned}$$

Thus,

$$\pi_{N-1} \leq \frac{|\Gamma_1|}{\delta_*^{N-1}} \leq \pi_{N-1} S_{f^*} \Lambda^{N-1}. \quad (4.3)$$

## 4.2 Improvements on the lower bound of the blow-up time

In the proof of Theorem 1.6, Lemma 2.3 is used to estimate the right hand side of (3.14). In this section, thanks to the extra geometric assumption on  $\Gamma_1$ , we can improve Lemma 2.3 and therefore obtain better lower bound estimates of the blow-up time. Depending on the dimension  $N \geq 3$  or  $N = 2$ , the following Lemma 4.4 and Lemma 4.5 improve Lemma 2.3 respectively.

**Lemma 4.4.** *Let  $\Omega$  and  $\Gamma_1$  be the same as in (1.1), if the dimension  $N \geq 3$  and  $\Gamma_1$  is ball-comparable with  $\Lambda$  and  $S_{f^*}$  defined in Definition 4.1, then there exists  $C = C(N)$  such that for any  $x \in \partial\Omega$  and  $t \geq 0$ ,*

$$\int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) dS(y) d\tau \leq C S_{f^*} \Lambda |\Gamma_1|^{\frac{1}{N-1}}. \quad (4.4)$$

*Proof.* We adopt the same notations  $y^*$ ,  $\delta_*$ ,  $\Lambda$ ,  $f^*$ ,  $S_{f^*}$  as in Definition 4.1. First by noticing

$$\int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) dS(y) d\tau = \int_0^t \int_{\Gamma_1} \Phi(x - y, \tau) dS(y) d\tau,$$

it suffices to estimate the latter one. By taking advantage of the condition (2) in Definition 4.1, there exists  $C = C(N)$  such that for any  $x \in \partial\Omega$ ,  $t > 0$  and  $\tau \in (0, t)$ ,

$$\begin{aligned}
\int_{\Gamma_1} \Phi(x - y, \tau) dS(y) &= C \tau^{-N/2} \int_{\Gamma_1} e^{-|x-y|^2/(4\tau)} dS(y) \\
&\leq C \tau^{-N/2} \int_{\tilde{B}(\overline{y^*}, \Lambda\delta_*)} e^{-|x-(\tilde{y}, f^*(\tilde{y}))|^2/(4\tau)} \sqrt{1 + |Df^*(\tilde{y})|^2} d\tilde{y} \\
&\leq C S_{f^*} \tau^{-N/2} \int_{\tilde{B}(\overline{y^*}, \Lambda\delta_*)} e^{-|\bar{x}-\tilde{y}|^2/(4\tau)} d\tilde{y} \\
&\leq C S_{f^*} \tau^{-N/2} \int_{\tilde{B}(\overline{y^*}, \Lambda\delta_*)} e^{-|\overline{y^*}-\tilde{y}|^2/(4\tau)} d\tilde{y},
\end{aligned} \quad (4.5)$$

where the last inequality is due to the symmetry of  $\tilde{B}(\overline{y^*}, \Lambda\delta_*)$  around  $\overline{y^*}$ . Now using polar coordinates, it follows from (4.5) that

$$\begin{aligned}
\int_{\Gamma_1} \Phi(x - y, \tau) dS(y) &\leq C S_{f^*} \tau^{-N/2} \int_{\tilde{B}(\bar{0}, \Lambda\delta_*)} e^{-|\tilde{y}|^2/(4\tau)} d\tilde{y} \\
&= C S_{f^*} \tau^{-N/2} \int_0^{\Lambda\delta_*} \rho^{N-2} e^{-\rho^2/(4\tau)} d\rho.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^t \int_{\Gamma_1} \Phi(x-y, \tau) dS(y) d\tau &\leq C S_{f^*} \int_0^t \int_0^{\Lambda\delta_*} \tau^{-N/2} \rho^{N-2} e^{-\rho^2/(4\tau)} d\rho d\tau \\
&= C S_{f^*} \int_0^{\Lambda\delta_*} \int_0^t \tau^{-N/2} \rho^{N-2} e^{-\rho^2/(4\tau)} d\tau d\rho \\
&= C S_{f^*} \int_0^{\Lambda\delta_*} \int_{\frac{\rho^2}{4t}}^\infty s^{\frac{N}{2}-2} e^{-s} ds d\rho,
\end{aligned} \tag{4.6}$$

where the change of variable  $\tau \rightarrow s \triangleq \rho^2/(4\tau)$  is used in the last equality.

Thanks to  $N \geq 3$ ,  $s^{N/2-2}$  is integrable near 0, so

$$\int_{\frac{\rho^2}{4t}}^\infty s^{\frac{N}{2}-2} e^{-s} ds \leq \int_0^\infty s^{\frac{N}{2}-2} e^{-s} ds = C.$$

Then it follows from (4.6) that

$$\int_0^t \int_{\Gamma_1} \Phi(x-y, \tau) dS(y) d\tau \leq C S_{f^*} \Lambda\delta_*.$$

Now recalling the relation (4.3), we finish the proof.  $\square$

**Lemma 4.5.** *Let  $\Omega$  and  $\Gamma_1$  be the same as in (1.1), if the dimension  $N = 2$  and  $\Gamma_1$  is ball-comparable with  $\delta_*$ ,  $\Lambda$  and  $S_{f^*}$  defined in Definition 4.1 and  $\Lambda\delta_* \leq 1/2$ , then there exists a universal constant  $C > 0$  such that for any  $x \in \partial\Omega$  and  $t \in [0, 1]$ ,*

$$\int_0^t \int_{\Gamma_1} \Phi(x-y, t-\tau) dS(y) d\tau \leq C S_{f^*} \Lambda |\Gamma_1| \ln \left( \frac{2 S_{f^*}}{|\Gamma_1|} \right). \tag{4.7}$$

*Proof.* We follow the proof of Lemma 4.4 identically until (4.6):

$$\int_0^t \int_{\Gamma_1} \Phi(x-y, \tau) dS(y) d\tau \leq C S_{f^*} \int_0^{\Lambda\delta_*} \int_{\frac{\rho^2}{4t}}^\infty s^{\frac{N}{2}-2} e^{-s} ds d\rho, \tag{4.6'}$$

where  $C$  is a constant depending only on the dimension and therefore is a universal constant in the current case since  $N = 2$ . The situation after this step is different, since  $N = 2$  makes  $s^{N/2-2} = s^{-1}$  not integrable near 0. So the right hand side of (4.6') has to be estimated more carefully.

Noticing  $\Lambda\delta_* \leq 1/2$  in the assumption, so  $\rho^2/4 \leq \frac{1}{16} < 1$  when  $0 \leq \rho \leq \Lambda\delta_*$ . Hence for any  $t \leq 1$ ,

$$\begin{aligned}
\int_{\frac{\rho^2}{4t}}^\infty s^{\frac{N}{2}-2} e^{-s} ds &= \int_{\frac{\rho^2}{4t}}^\infty s^{-1} e^{-s} ds \\
&\leq \int_{\frac{\rho^2}{4}}^\infty s^{-1} e^{-s} ds \\
&= \int_{\frac{\rho^2}{4}}^1 s^{-1} e^{-s} ds + \int_1^\infty s^{-1} e^{-s} ds \\
&\leq \int_{\frac{\rho^2}{4}}^1 s^{-1} ds + \int_1^\infty e^{-s} ds \\
&< 3 - 2 \ln(\rho).
\end{aligned}$$

Thus,

$$\begin{aligned} \int_0^{\Lambda \delta_*} \int_{\frac{\rho^2}{4t}}^{\infty} s^{\frac{N}{2}-2} e^{-s} ds d\rho &\leq \int_0^{\Lambda \delta_*} 3 - 2 \ln(\rho) d\rho \\ &= 5\Lambda \delta_* - 2\Lambda \delta_* \ln(\Lambda \delta_*). \end{aligned} \quad (4.8)$$

Taking advantage of the assumption  $\Lambda \delta_* \leq 1/2$  again, it follows from (4.8) that

$$\int_0^{\Lambda \delta_*} \int_{\frac{\rho^2}{4t}}^{\infty} s^{\frac{N}{2}-2} e^{-s} ds d\rho \leq C \Lambda \delta_* \ln \left( \frac{1}{\Lambda \delta_*} \right) \quad (4.9)$$

for some universal constant  $C$ .

Noticing that the relation (4.3) for  $N = 2$  becomes

$$\delta_* \leq \frac{|\Gamma_1|}{2} \quad \text{and} \quad \frac{1}{\Lambda \delta_*} \leq \frac{2 S_{f^*}}{|\Gamma_1|}, \quad (4.10)$$

then it follows from (4.9) that

$$\int_0^{\Lambda \delta_*} \int_{\frac{\rho^2}{4t}}^{\infty} s^{\frac{N}{2}-2} e^{-s} ds d\rho \leq C \Lambda |\Gamma_1| \ln \left( \frac{2 S_{f^*}}{|\Gamma_1|} \right). \quad (4.11)$$

Combining (4.6') and (4.11), we obtain

$$\int_0^t \int_{\Gamma_1} \Phi(x - y, \tau) dS(y) d\tau \leq C S_{f^*} \Lambda |\Gamma_1| \ln \left( \frac{2 S_{f^*}}{|\Gamma_1|} \right).$$

□

Now we start to justify Theorem 1.7 and Theorem 1.8.

**Proof of Theorem 1.7.** We follow the proof of Theorem 1.6 identically but applying Lemma 4.4 instead of Lemma 2.3 to estimate the right hand side of (3.14). Comparing these two lemmas, firstly it follows from Lemma 2.3 that for any  $(x, t) \in \partial\Omega \times [0, 1]$ ,

$$\int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) dS(y) d\tau \leq C^* |\Gamma_1|^\alpha. \quad (2.7')$$

Secondly Lemma 4.4 implies that for any  $(x, t) \in \partial\Omega \times [0, 1]$ ,

$$\int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) dS(y) d\tau \leq C_1 S_{f^*} \Lambda |\Gamma_1|^{\frac{1}{N-1}} \quad (4.4')$$

for some constant  $C_1 = C_1(N)$ . Observing these two estimates, we can regard the right hand side of (4.4') to play the same role as the right hand side of (2.7'). Then by replacing  $C^* |\Gamma_1|^\alpha$  with  $C_1 S_{f^*} \Lambda |\Gamma_1|^{\frac{1}{N-1}}$  in (3.22), we obtain an analogous result to (3.22):

$$T^* > \frac{b_1 M_0^{1-q}}{20(q-1) C_1 S_{f^*} \Lambda |\Gamma_1|^{\frac{1}{N-1}}} - \frac{3q}{10(q-1)}. \quad (3.22')$$

Finally we finish the proof by choosing

$$C = \frac{b_1}{20 C_1}$$

in (1.12).  $\square$

**Proof of Theorem 1.8.** Again we follow the proof of Theorem 1.6 identically but applying Lemma 4.5 instead of Lemma 2.3 to estimate the right hand side of (3.14). This time Lemma 4.5 implies that for any  $(x, t) \in \partial\Omega \times [0, 1]$ ,

$$\int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) dS(y) d\tau \leq C_2 S_{f^*} \Lambda |\Gamma_1| \ln \left( \frac{2 S_{f^*}}{|\Gamma_1|} \right), \quad (4.7')$$

for some universal constant  $C_2$ . As a result, we can regard the right hand side of (4.7') to play the same role as the right hand side of (2.7'). Then by replacing  $C^* |\Gamma_1|^\alpha$  with  $C_2 S_{f^*} \Lambda |\Gamma_1| \ln \left( \frac{2 S_{f^*}}{|\Gamma_1|} \right)$  in (3.22), we obtain an analogous result to (3.22):

$$T^* > \frac{b_1 M_0^{1-q}}{20(q-1) C_2 S_{f^*} \Lambda |\Gamma_1| \ln \left( \frac{2 S_{f^*}}{|\Gamma_1|} \right)} - \frac{3q}{10(q-1)}. \quad (3.22'')$$

Finally we finish the proof by choosing

$$C = \frac{b_1}{20 C_2}$$

in (1.13).  $\square$

## 5 Numerical Simulations

It has been discussed in the introduction that there is a big gap between the upper bound estimate and the lower bound estimate of  $T^*$  as  $|\Gamma_1| \rightarrow 0$ , so we are wondering which bound is more reliable. In Section 4 of [19], some numerical simulations are performed and the results indicated that the lower bound of  $T^*$  was in the order of  $|\Gamma_1|^{-1}$  as  $|\Gamma_1| \rightarrow 0$ . Combining with the upper bound given in Theorem 1.4 which is also in the order of  $|\Gamma_1|^{-1}$ , it implied that the blow-up time  $T^*$  should be comparable to  $|\Gamma_1|^{-1}$  as  $|\Gamma_1| \rightarrow 0$ . However, in order to keep accuracy of the schemes, the numerical simulations in [19] only calculated until  $u$  reached 10 (from a relatively small initial data  $u_0 \equiv 0.05$ ). This restriction may not give a good estimate for  $T^*$  since it drops the time after  $u$  reaches 10. So the purpose of this section is to provide a more robust way to estimate the lower bound of  $T^*$  numerically.

Let  $u$  be the maximal solution to (1.1), in order to avoid losing accuracy when  $u$  becomes large, we consider the transformation  $v = u^{1-q}$ , which converts the large function  $u$  into a small function  $v$  as  $t \rightarrow T^*$ . Thus the task becomes to estimate the lower bound of the time  $T^*$  for  $v$  to vanish at some point. By direct calculations, it follows from (1.1) that

$$\begin{cases} v_t(x, t) = \Delta v(x, t) - \frac{q}{q-1} |\nabla v(x, t)|^2 / v(x, t) & \text{in } \Omega \times (0, T], \\ \frac{\partial v}{\partial n}(x, t) = 1 - q & \text{on } \Gamma_1 \times (0, T], \\ \frac{\partial v}{\partial n}(x, t) = 0 & \text{on } \Gamma_2 \times (0, T], \\ v(x, 0) = v_0(x) \triangleq u_0^{1-q}(x) & \text{in } \Omega, \end{cases} \quad (5.1)$$

where  $q$  and  $u_0$  are the same as those in (1.1).

Since  $v$  can be very close to 0 or even negative during the numerical simulation when  $t$  is close to  $T^*$ , the denominator of the term  $|\nabla v(x, t)|^2 / v(x, t)$  should be adjusted to  $\sqrt{v^2(x, t) + \epsilon^2}$  for small  $\epsilon$ . Hence it is



attempting to consider the following problem for  $w_\epsilon$ :

$$\begin{cases} (w_\epsilon)_t(x, t) = \Delta w_\epsilon(x, t) - \frac{q}{q-1} |\nabla w_\epsilon(x, t)|^2 / \sqrt{w_\epsilon^2(x, t) + \epsilon^2} & \text{in } \Omega \times (0, T], \\ \frac{\partial w_\epsilon}{\partial n}(x, t) = 1 - q & \text{on } \Gamma_1 \times (0, T], \\ \frac{\partial w_\epsilon}{\partial n}(x, t) = 0 & \text{on } \Gamma_2 \times (0, T], \\ w_\epsilon(x, 0) = w_0(x) \triangleq u_0^{1-q}(x) & \text{in } \Omega. \end{cases} \quad (5.2)$$

But this  $w_\epsilon$  is larger than  $v$  and therefore it will reach 0 after  $T^*$ . So if we approximate the time when  $w_\epsilon$  reaches 0, it may not be a lower bound for  $T^*$ .

To handle this problem, we actually study the following problem for  $v_\epsilon$ :

$$\begin{cases} (v_\epsilon)_t(x, t) = \Delta v_\epsilon(x, t) - \frac{q}{q-1} |\nabla v_\epsilon(x, t)|^2 / \sqrt{v_\epsilon^2(x, t) + \frac{\epsilon^2}{4}} - \epsilon & \text{in } \Omega \times (0, T], \\ \frac{\partial v_\epsilon}{\partial n}(x, t) = 1 - q - \epsilon & \text{on } \Gamma_1 \times (0, T], \\ \frac{\partial v_\epsilon}{\partial n}(x, t) = -\epsilon & \text{on } \Gamma_2 \times (0, T], \\ v_\epsilon(x, 0) = v_0(x) - \epsilon = u_0^{1-q}(x) - \epsilon & \text{in } \Omega, \end{cases} \quad (5.3)$$

where  $q$  and  $u_0$  are the same as those in (1.1). Let  $T^*$  be the time for  $v$  to reach 0 in (5.1) and let  $T_\epsilon^*$  be the time for  $v_\epsilon$  to reach 0 in (5.3), then for any  $T < \min\{T^*, T_\epsilon^*\}$ , we claim

$$\min_{(x,t) \in \overline{\Omega} \times [0, T]} (v - v_\epsilon)(x, t) > \frac{1}{2} \epsilon. \quad (5.4)$$

In fact, if (5.4) is not true, let  $t_0$  be the first time for  $(v - v_\epsilon)$  to reach  $\epsilon/2$ , then  $t_0 > 0$ , since  $(v - v_\epsilon)(x, 0) = \epsilon$  for any  $x \in \overline{\Omega}$ . Moreover, there exists  $x_0 \in \overline{\Omega}$  such that

$$(v - v_\epsilon)(x_0, t_0) = \frac{\epsilon}{2} = \min_{(x,t) \in \overline{\Omega} \times [0, t_0]} (v - v_\epsilon)(x, t). \quad (5.5)$$

Then there are two situations.

- Case 1:  $x_0 \in \partial\Omega$ . In this case, on the one hand, (5.5) implies that  $\frac{\partial(v-v_\epsilon)}{\partial n}(x_0, t_0) \leq 0$ . On the other hand, it follows from (5.1) and (5.3) that

$$\frac{\partial(v - v_\epsilon)}{\partial n}(x_0, t_0) = \epsilon > 0.$$

Thus there is a contradiction.

- Case 2:  $x_0 \in \Omega$ . In this case, (5.5) implies that  $\nabla v(x_0, t_0) = \nabla v_\epsilon(x_0, t_0)$  and

$$v(x_0, t_0) = v_\epsilon(x_0, t_0) + \frac{\epsilon}{2} > \sqrt{v_\epsilon^2(x_0, t_0) + \frac{\epsilon^2}{4}}.$$

Then we can derive a contradiction by comparing the first equations in (5.1) and (5.3).

Hence (5.4) must hold. Then it follows from (5.4) that  $T_\epsilon^* < T^*$  and

$$\inf_{(x,t) \in \overline{\Omega} \times [0, T_\epsilon^*]} (v - v_\epsilon)(x, t) \geq \frac{1}{2} \epsilon.$$

So any lower bound  $T_0$  for  $T_\epsilon^*$  automatically becomes a lower bound for  $T^*$ .

Now it is time to perform the numerical simulations. Let  $\epsilon = 10^{-8}$  and  $m_1 : [0, T^*) \rightarrow \mathbb{R}$  be defined as

$$m_1(t) = \min_{x \in \Omega} v_\epsilon(x, t),$$

we apply the Finite Difference Method to (5.3). The following are a summary of the parameters we choose.

- (1) 2-D Cases. Unit square:  $(x, y) \in [0, 1] \times [0, 1]$ , space step size  $h=1/40$ , time step size  $k = 0.2h^2$ ,  $\Gamma_1$  is located on the middle part of the side  $y = 1$  and the length  $|\Gamma_1|$  is decreasing from 20/40 to 3/40 as shown in Table 1 and Table 2.
- (2) 3-D Cases. Unit cubic:  $(x, y, z) \in [0, 1] \times [0, 1] \times [0, 1]$ , space step size  $h=1/10$ , time step size  $k = 0.1h^2$ ,  $\Gamma_1$  is located on the middle part of the face  $z = 1$  and the area  $|\Gamma_1|$  is decreasing from 49/100 to 9/100 as shown in Table 3 and Table 4.

Results for 2-D cases:

- Let  $q = 2$  and initial data  $u_0(x) \equiv 0.05$ . Table 1 denotes the first time  $T_0$  for  $m_1$  to be below  $10^{-4}$ .

$ \Gamma_1 $	$T_0$	Order	$m_1(T_0)$
20/40	35.39		-5.90E-3
10/40	72.75	1.04	-2.20E-2
5/40	149.46	1.04	-4.30E-1
3/40	253.22	1.04	-2.31E-2

Table 1: 2 Dimension,  $q = 2$

- Let  $q = 3$  and initial data  $u_0(x) \equiv 0.05$ . Table 2 denotes the first time  $T_0$  for  $m_1$  to be below  $10^{-4}$ .

$ \Gamma_1 $	$T_0$	Order	$m_1(T_0)$
20/40	394.60		-8.21E-2
10/40	791.77	1.00	-2.36E-2
5/40	1588.35	1.00	-3.07E-2
3/40	2652.08	1.00	-2.64E-2

Table 2: 2-D,  $q = 3$

Results for 3-D cases:

- Let  $q = 2$  and initial data  $u_0(x) \equiv 0.05$ . Table 3 denotes the first time  $T_0$  for  $m_1$  to be below  $10^{-4}$ .

$ \Gamma_1 $	$T_0$	Order	$m_1(T_0)$
49/100	36.19		-4.24E-2
25/100	72.19	1.03	-1.32E-3
16/100	113.93	1.02	-2.77E-2
9/100	205.30	1.02	-1.13E-3

Table 3: 3-D,  $q = 2$

- Let  $q = 3$  and initial data  $u_0(x) \equiv 0.05$ . Table 4 denotes the first time  $T_0$  for  $m_1$  to be below  $10^{-4}$ .

From these tables, we have two comments.

$ \Gamma_1 $	$T_0$	Order	$m_1(T_0)$
49/100	402.75		-6.00E-1
25/100	791.06	1.00	-1.49
16/100	1237.50	1.00	-1.87E-3
9/100	2203.30	1.00	-3.08

Table 4: 3-D,  $q = 3$

- (1) Firstly, it is evident that  $v_\epsilon$  becomes negative immediately after it is less than  $10^{-4}$ . Thus, the time  $T_0$  should be a good estimate for  $T_\epsilon^*$ . In addition, during the simulation,  $v_\epsilon$  is set to be larger than  $10^{-4}$ , which is much larger than  $\epsilon = 10^{-8}$ . As a result,  $v_\epsilon$  should be very close to  $v$  and therefore  $T_\epsilon^*$  is close to  $T^*$ . Thus  $T_0$  should be close to  $T^*$  as well.
- (2) One can see from any of these tables that the order of  $T_0$  is about  $|\Gamma_1|^{-1}$  as  $|\Gamma_1| \rightarrow 0$ . So we still conjecture that the order of the blow-up time  $T^*$  for (1.1) is  $|\Gamma_1|^{-1}$ , since an upper bound of  $T^*$  has been shown to be comparable to  $|\Gamma_1|^{-1}$  in Theorem 1.4.
- (3) Actually if we compare the numerical results in the above tables to the results in Section 4 of [19], they are very close. But the schemes in this section calculate until  $u = 10^{4/(q-1)}$  (or equivalently  $v = 10^{-4}$ ), which is much larger than  $u = 10$  in [19] when  $q = 2$  or  $3$ . So the results in this section are more reliable.

## A Representation formula from any time

In Corollary 3.9 of [19], it derived the Representation formula (1.16), where the initial time is 0 and the initial data is  $u(\cdot, 0) = u_0(\cdot)$ . Now for any  $T \in (0, T^*)$ , we are asking that if regarding  $T$  to be the initial time and  $u(\cdot, T)$  to be the initial data, then is there a Representation formula similar to (1.16)? It seems trivial, but we should be careful, since Corollary 3.9 in [19] deals with  $C^1(\overline{\Omega})$  initial data but  $u(\cdot, T)$  is only in  $C^2(\Omega) \cap C(\overline{\Omega})$ . The next lemma and its proof show that by approximation, we can overcome this difficulty to obtain the expected Representation formula.

**Lemma A.1.** *Let  $T^*$  be the maximal existence time and  $u$  be the maximal solution of (1.1), then for any  $T \in [0, T^*)$  and  $(x, t) \in \partial\Omega \times [0, T^* - T)$ ,*

$$\begin{aligned}
u(x, T + t) &= 2 \int_{\Omega} \Phi(x - y, t) u(y, T) dy - 2 \int_0^t \int_{\partial\Omega} D_y [\Phi(x - y, t - \tau)] \cdot \vec{n}(y) u(y, T + \tau) dS(y) d\tau \\
&\quad + 2 \int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) u^q(y, T + \tau) dS(y) d\tau.
\end{aligned} \tag{A.1}$$

*Proof.* When  $T = 0$ , (A.1) is just the Representation formula (1.16) which has been proven in [19]. Now let  $T > 0$  and we intend to verify (A.1) which is a Representation formula with initial time  $T$  and initial data  $u(\cdot, T)$ .

Defining  $v : \overline{\Omega} \times [0, T^* - T) \rightarrow \mathbb{R}$  by  $v(x, t) = u(x, T + t)$ , then  $v \in C^{2,1}(\Omega \times (0, T^* - T)) \cap C(\overline{\Omega} \times [0, T^* - T))$  and satisfies

$$\begin{cases} v_t(x, t) = \Delta v(x, t) & \text{in } \Omega \times (0, T^* - T), \\ \frac{\partial v}{\partial n}(x, t) = u^q(x, T + t) & \text{on } \Gamma_1 \times (0, T^* - T), \\ \frac{\partial v}{\partial n}(x, t) = 0 & \text{on } \Gamma_2 \times (0, T^* - T), \\ v(x, 0) = u(x, T) & \text{in } \Omega. \end{cases} \tag{A.2}$$

Note that (A.2) is a linear problem in  $v$ , since  $u$  is a fixed function.

Now we continuously extend  $u(\cdot, T)$  to  $\mathbb{R}^N$  and still denote it to be  $u(\cdot, T)$ . Then for any  $j \geq 1$ , we choose the standard mollifier  $\eta_j$  such that

$$\max_{x \in \overline{\Omega}} |g_j(x) - u(x, T)| \leq 1/j, \quad (\text{A.3})$$

where

$$g_j(x) \triangleq (\eta_j(\cdot) * u(\cdot, T))(x).$$

Since  $g_j \in C^1(\overline{\Omega})$ , it follows from Theorem B.4 in [19] that there exists  $v_j \in C^{2,1}(\Omega \times (0, T^* - T)) \cap C(\overline{\Omega} \times [0, T^* - T])$  such that

$$\begin{cases} (v_j)_t(x, t) = \Delta v_j(x, t) & \text{in } \Omega \times (0, T^* - T), \\ \frac{\partial v_j}{\partial n}(x, t) = u^q(x, T + t) & \text{on } \Gamma_1 \times (0, T^* - T), \\ \frac{\partial v_j}{\partial n}(x, t) = 0 & \text{on } \Gamma_2 \times (0, T^* - T), \\ v_j(x, 0) = g_j(x) & \text{in } \Omega. \end{cases} \quad (\text{A.4})$$

Taking advantage of  $g_j \in C^1(\overline{\Omega})$  again and similarly to (1.16), there exists a Representation formula for (A.4): for any  $(x, t) \in \partial\Omega \times [0, T^* - T]$ ,

$$\begin{aligned} v_j(x, t) &= 2 \int_{\Omega} \Phi(x - y, t) g_j(y) dy - 2 \int_0^t \int_{\partial\Omega} D_y[\Phi(x - y, t - \tau)] \cdot \vec{n}(y) v_j(y, \tau) dS(y) d\tau \\ &\quad + 2 \int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) u^q(y, T + \tau) dS(y) d\tau. \end{aligned} \quad (\text{A.5})$$

Let  $w_j = v_j - v$ , then  $w_j \in C^{2,1}(\Omega \times (0, T^* - T)) \cap C(\overline{\Omega} \times [0, T^* - T])$  and satisfies

$$\begin{cases} (w_j)_t(x, t) = \Delta w_j(x, t) & \text{in } \Omega \times (0, T^* - T), \\ \frac{\partial w_j}{\partial n}(x, t) = 0 & \text{on } \Gamma_1 \times (0, T^* - T), \\ \frac{\partial w_j}{\partial n}(x, t) = 0 & \text{on } \Gamma_2 \times (0, T^* - T), \\ w_j(x, 0) = g_j(x) - u(x, T) & \text{in } \Omega. \end{cases}$$

So it follows from the maximum principle and the Hopf lemma that for any  $(x, t) \in \overline{\Omega} \times [0, T^* - T]$ ,

$$|w_j(x, t)| \leq \max_{x \in \overline{\Omega}} |g_j(x) - u(x, T)| \leq 1/j.$$

Thus

$$|v_j(x, t) - v(x, t)| \leq 1/j, \quad \forall (x, t) \in \overline{\Omega} \times [0, T^* - T].$$

Now fixing any point  $(x, t) \in \partial\Omega \times [0, T^* - T]$  and let  $j \rightarrow \infty$  in (A.5), then it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned} v(x, t) &= 2 \int_{\Omega} \Phi(x - y, t) u(y, T) dy - 2 \int_0^t \int_{\partial\Omega} D_y[\Phi(x - y, t - \tau)] \cdot \vec{n}(y) v(y, \tau) dS(y) d\tau \\ &\quad + 2 \int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) u^q(y, T + \tau) dS(y) d\tau. \end{aligned}$$

Finally plugging in  $v(x, t) = u(x, T + t)$  and  $v(y, \tau) = u(y, T + \tau)$ , we obtain (A.1).  $\square$

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